

A Quantum Exactly Solvable Nonlinear Oscillator with quasi-Harmonic Behaviour

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Abstract

The quantum version of a non-linear oscillator, previously analyzed at the classical level, is studied. This is a problem of quantization of a system with position-dependent mass of the form $m = (1 + \lambda x^2)^{-1}$ and with a λ -dependent nonpolynomial rational potential. This λ -dependent system can be considered as a deformation of the harmonic oscillator in the sense that for $\lambda \rightarrow 0$ all the characteristics of the linear oscillator are recovered. Firstly, the λ -dependent Schrödinger equation is exactly solved as a Sturm-Liouville problem and the λ -dependent eigenenergies and eigenfunctions are obtained for both $\lambda > 0$ and $\lambda < 0$. The λ -dependent wave functions appear as related with a family of orthogonal polynomials that can be considered as λ -deformations of the standard Hermite polynomials. In the second part, the λ -dependent Schrödinger equation is solved by using the Schrödinger factorization method, the theory of intertwined Hamiltonians and the property of shape invariance as an approach. Finally, the new family of orthogonal polynomials is studied. We prove the existence of a λ -dependent Rodrigues formula, a generating function and λ -dependent recursion relations between polynomials of different orders.

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1 Introduction

The nonlinear differential equation

$$(1 + \lambda x^2) \ddot{x} - (\lambda x) \dot{x}^2 + \alpha^2 x = 0, \quad \lambda > 0, \quad (1)$$

was studied by Mathews and Lakshmanan in [1] (see also [2]) as an example of a non-linear oscillator (notice α^2 was written just as α in the original paper); the most remarkable property is the existence of solutions of the form

$$x = A \sin(\omega t + \phi),$$

with the following additional restriction linking frequency and amplitude

$$\omega^2 = \frac{\alpha^2}{1 + \lambda A^2}.$$

That is, the equation (1) represents a non-linear oscillator with periodic solutions that were qualified as having a “simple harmonic form”. The authors also proved that (1) is obtainable from the Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda x^2} \right) (\dot{x}^2 - \alpha^2 x^2) \quad (2)$$

which they considered as the one-dimensional analogue of the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left(\frac{1}{1 + \lambda \phi^2} \right) (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2),$$

appearing in some nonpolynomial models of quantum field theory.

The nonlinear equation (1) is therefore an interesting example of a system with nonlinear quasi-harmonic oscillations. Recently, it has been proved [3] that this particular nonlinear system can be generalized to the two-dimensional case, and even to the n -dimensional case and that these higher dimensional systems are superintegrable with $2n - 1$ quadratic constants of motion. Moreover, we point out that a geometric interpretation of the higher-dimensional systems was proposed in relation with the dynamics on spaces of constant curvature. It was also proved the existence of a related λ -dependent isotonic oscillator and that the two-dimensional oscillator, previously studied in [3], admits a superintegrable modification that corresponds to the λ -dependent version of the Smorodinski-Winternitz system [4, 5]. In fact this means that the deformation introduced by the parameter λ modifies the Hamilton-Jacobi equation but preserves the existence of a multiple separability.

On the other hand, Biswas *et al* studied in 1973 [6] the ground state as well as some excited energy levels of the generalized anharmonic oscillator defined by the Hamiltonian $H_m = -d^2/dx^2 + x^2 + \lambda x^{2m}$, $m = 2, 3, \dots$, and then they proposed the use of similar techniques for the analysis of the Schrödinger equation involving the potential $\lambda(x^2/(1 + gx^2))$. Since then, this nonpolynomial potential has been extensively studied by many authors [7]–[27] from different viewpoints and by making use of different approaches. In many cases the term $x^2/(1 + gx^2)$ was introduced as a perturbation of an initial harmonic oscillator; that is, the potential to be solved was not $x^2/(1 + gx^2)$ by itself but $x^2 + \lambda x^2/(1 + gx^2)$ (exact analytical solutions have only been found for certain very particular values of the parameters λ and g , see e.g., Refs. [12]–[24]). It is important to note that,

in most of these cases, the derivative part of the Schrödinger equation was the standard one, that is, the equation arising from a Hamiltonian H with a quadratic term of the form $(1/2)p^2$ and leading to a derivative term of the form $-d^2/dx^2$, that is

$$\left[-\frac{d^2}{dx^2} + x^2 + \lambda \frac{x^2}{(1+gx^2)}\right] \Psi_n = E_n \Psi_n$$

or the corresponding two or three-dimensional versions involving the Laplace operator in \mathbb{E}^2 or \mathbb{E}^3 .

The important point is that, in the Lagrangian (2), the parameter λ is present not only in potential $x^2/(1+\lambda x^2)$ but also in the kinetic term. So, this nonlinear oscillator must be considered as a particular case of a system with a position-dependent effective mass [28]. In fact, it is known the existence of physical systems for which the mass of a nonrelativistic quantum particle may vary with its position (they correspond, for example, to the motion in an external field in a crystal, electronic properties of semiconductors, liquid crystals, etc.) and in recent years special interest has been drawn to generalize the standard methods of solving the Schrödinger equation to this new class of systems [29]-[39]. For example, some techniques have been proposed to convert the Schrödinger equation with a position-dependent mass into a Schrödinger equation with a constant mass (see [39] and references therein). Nevertheless there is an important problem at the starting level of quantization (that is, transition from the classical system to the quantum one) since if the mass m becomes a spatial function, $m = m(x)$, then the quantum version of the mass no longer commutes with the momentum. Therefore, different forms of presenting the kinetic term in the Hamiltonian H , as for example

$$T = \frac{1}{4} \left[\frac{1}{m(x)} p^2 + p^2 \frac{1}{m(x)} \right], \quad T = \frac{1}{2} \left[\frac{1}{\sqrt{m(x)}} p^2 \frac{1}{\sqrt{m(x)}} \right], \quad T = \frac{1}{2} \left[p \frac{1}{m(x)} p \right],$$

are equivalent at the classical level but they lead to different nonequivalent Schrödinger equations.

In fact, the quantum version of Mathews and Lakshmanan oscillator was studied in [7] by making use of the following Hamiltonian

$$H = \frac{1}{2} \left[\frac{1}{2} \{p^2, (1 - gx^2)\} + \frac{k x^2}{(1 - gx^2)} \right],$$

where the notation $\{A, B\} = AB + BA$ is used. More recently another different approach has been discussed in Ref. [40]. It was proposed a rule for the transition from the classical system to the quantum one that was obtained from an analysis of the geometric properties of the function representing the classical kinetic energy. One of the advantages of this procedure was that the quantum system so obtained can be studied by using a factorization method [41]-[46] and that it is even endowed with the property of shape invariance [47]-[58].

The main objective of this article is to continue with the analysis of the quantum version of this particular nonlinear oscillator and to present a detailed study not only of the central question but also of some questions related with it. The main points to be discussed in this paper can be summarized in the following four points:

- Analysis of the transition from the classical λ -dependent system to the quantum one.

This is a problem of quantization of a system with position-dependent mass and in this particular case the quantization procedure to be analyzed is related with the existence, at the classical level, of a Killing vector and also of a λ -dependent invariant measure.

- Analysis and exact resolution of the λ -dependent Schrödinger equation as a Sturm-Liouville problem.

In fact, not just one but two different Sturm-Liouville problems are obtained: one for λ positive and other for λ negative. This also means a discussion of the properties of the energy levels and of wave functions depending of the sign of λ .

- Existence of a family of λ -dependent orthogonal polynomials.

The λ -dependent system we study can be considered as a deformation of the harmonic oscillator in the sense that for $\lambda \rightarrow 0$ all the characteristics of the linear oscillator must be recovered. The λ -dependent wave functions will appear as related with a family of polynomials that can be considered as λ -deformations of the standard Hermite polynomials. This means that the resolution of the quantum problem leads in a natural way to the study of a new family of orthogonal polynomials.

- Resolution of the λ -dependent Schrödinger equation by using the Schrödinger factorization method as an approach.

This means a discussion of this λ -dependent quantum problem as a shape-invariant problem. Many properties of the λ -dependent orthogonal polynomials, as the generating function and some recursion relations, are obtained as a byproduct of this factorization approach.

In more detail, the plan of the article is as follows: In Sec. II we first discuss the quantization of the classical system and then we solve the λ -dependent Schrödinger equation; we analyze the characteristics of the solutions and we prove that they depend on a family of polynomials related with the Hermite polynomials; in the last subsection we prove the orthogonality of these new polynomials. Sec. III is devoted to review the main characteristics of the Schrödinger factorization formalism and in Sec. IV we apply this method to the λ -dependent nonlinear oscillator and we obtain, using the operators A and A^+ , the energies E_n and the wave functions Ψ_n . In Sec. V we study the main properties of the λ -dependent “deformed Hermite” polynomials; in particular we prove the existence of a λ -dependent Rodrigues formula, of a generating function and of λ -dependent recursion relations between polynomials of different orders. Finally, in Sec. VI we discuss the results and make some final comments.

2 λ -dependent Schrödinger equation

2.1 Quantization and Schrödinger equation

In the general case of a system with position-dependent effective mass the transition from the classical system to the quantum one is a difficult problem because of the ambiguities in the order of the factors. Nevertheless, in the particular case of the Lagrangian (2), it was proved in Ref. [40] that the structure of the kinetic term suggests us a very clear and direct procedure for the quantization.

Let us begin by considering the one-dimensional free-particle motion characterized by the λ -dependent kinetic term $T(\lambda)$ as a Lagrangian

$$L(x, v_x; \lambda) = T_\lambda(x, v_x) = \frac{1}{2} \left(\frac{v_x^2}{1 + \lambda x^2} \right), \quad (3)$$

and the following nonlinear equation

$$(1 + \lambda x^2) \ddot{x} - (\lambda x) \dot{x}^2 = 0,$$

where we point out that we admit λ can take both positive and negative values; of course it is clear that for $\lambda < 0$, $\lambda = -|\lambda|$, the function (and the associated dynamics) will have a singularity at $1 - |\lambda| x^2 = 0$ and we shall restrict the study of the dynamics to the interior of the interval $x^2 < 1/|\lambda|$ where the kinetic energy function T_λ is positive definite.

The function T_λ is invariant under the action of the vector field $X(\lambda)$ given by

$$X_\lambda(x) = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x},$$

in the sense that we have

$$X_\lambda^t(T_\lambda) = 0,$$

where X_λ^t denotes the natural lift to the velocity phase space $\mathbb{R} \times \mathbb{R}$ (the tangent bundle in differential geometric terms) of the vector field X_λ ,

$$X_\lambda^t = \sqrt{1 + \lambda x^2} \frac{\partial}{\partial x} + \left(\frac{\lambda x v_x}{\sqrt{1 + \lambda x^2}} \right) \frac{\partial}{\partial v_x}.$$

In differential geometric terms this property means that the vector field X_λ is a Killing vector of the one-dimensional metric

$$ds_\lambda^2 = \left(\frac{1}{1 + \lambda x^2} \right) dx^2.$$

Moving to the quantum setting, we see that this property suggests the idea of working with functions and linear operators defined on the space obtained by considering the real line \mathbb{R} endowed with the measure $d\mu_\lambda$ given by

$$d\mu_\lambda = \left(\frac{1}{\sqrt{1 + \lambda x^2}} \right) dx, \quad (4)$$

which is (up to a factor) the only measure invariant under X_λ [40]. This means that the operator P_x representing the linear momentum must be self-adjoint not in the standard space $L^2(\mathbb{R})$ but in the space $L^2(\mathbb{R}, d\mu_\lambda)$.

Let us start our study with the classical Hamiltonian of the λ -dependent oscillator [3]

$$H = \left(\frac{1}{2m} \right) P_x^2 + \left(\frac{1}{2} \right) g \left(\frac{x^2}{1 + \lambda x^2} \right), \quad P_x = \sqrt{1 + \lambda x^2} p_x, \quad g = m\alpha^2. \quad (5)$$

Figures I and II show the form of the potential $V_\lambda(x)$ for several values of λ ($\lambda < 0$ in Figure I and $\lambda > 0$ in Figure II).

As it has been pointed out in [40], the generator of the infinitesimal 'translation' symmetry $\sqrt{1+\lambda x^2} d/dx$ is skew-selfadjoint in the space $L^2(\mathbb{R}, d\mu_\lambda)$ and therefore the transition from the classical system to the quantum one is given by defining the operator

$$P_x = -i\hbar \sqrt{1+\lambda x^2} \frac{d}{dx},$$

so that

$$(1+\lambda x^2) p_x^2 \rightarrow -\hbar^2 \left(\sqrt{1+\lambda x^2} \frac{d}{dx} \right) \left(\sqrt{1+\lambda x^2} \frac{d}{dx} \right),$$

in such a way that the quantum version of the Hamiltonian (5) becomes

$$\hat{H} = -\frac{\hbar^2}{2m} (1+\lambda x^2) \frac{d^2}{dx^2} - \left(\frac{\hbar^2}{2m} \right) \lambda x \frac{d}{dx} + \left(\frac{1}{2} \right) g \left(\frac{x^2}{1+\lambda x^2} \right).$$

Let us consider the following quantum Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} (1+\lambda x^2) \frac{d^2}{dx^2} - \left(\frac{\hbar^2}{2m} \right) \lambda x \frac{d}{dx} + \left(\frac{1}{2} \right) m\alpha \left(\alpha + \frac{\hbar}{m} \lambda \right) \left(\frac{x^2}{1+\lambda x^2} \right), \quad (6)$$

where we have slightly modified the value of the parameter g that now is given by $g = m\alpha^2 + \lambda\hbar\alpha$; this is done in order that the notation used in this approach coincides with the one to be presented in the next section III. It is also convenient to simplify this function \hat{H} by introducing adimensional variables (y, Λ) defined by

$$x = \left(\sqrt{\frac{\hbar}{m\alpha}} \right) y, \quad \lambda = \left(\frac{m\alpha}{\hbar} \right) \Lambda,$$

in such a way that the following equality is satisfied

$$1 + \lambda x^2 = 1 + \Lambda y^2.$$

The Hamiltonian \hat{H} takes then the following form

$$\hat{H} = \left[-\frac{1}{2} (1 + \Lambda y^2) \frac{d^2}{dy^2} - \left(\frac{1}{2} \right) \Lambda y \frac{d}{dy} + \left(\frac{1}{2} \right) (1 + \Lambda) \left(\frac{y^2}{1 + \Lambda y^2} \right) \right] (\hbar\alpha), \quad (7)$$

and then the Schrödinger equation

$$\hat{H} \Psi = E \Psi, \quad E = e (\hbar\alpha),$$

reduces to the following adimensional form

$$\left[-\frac{1}{2} (1 + \Lambda y^2) \frac{d^2}{dy^2} - \left(\frac{1}{2} \right) \Lambda y \frac{d}{dy} + \left(\frac{1}{2} \right) (1 + \Lambda) \left(\frac{y^2}{1 + \Lambda y^2} \right) \right] \Psi = e \Psi,$$

which, after a small simplification, can be finally rewritten as follows

$$(1 + \Lambda y^2) \frac{d^2}{dy^2} \Psi + \Lambda y \frac{d}{dy} \Psi - (1 + \Lambda) \left(\frac{y^2}{1 + \Lambda y^2} \right) \Psi + (2e) \Psi = 0. \quad (8)$$

It is known that in the $\Lambda = 0$ case the asymptotic behaviour at the infinity suggest a factorization for the wave function. The idea is that a similar procedure can be applied to this Λ -dependent

equation. Let us first denote by Ψ_∞ the following function $\Psi_\infty = (1 + \Lambda y^2)^{-1/(2\Lambda)}$ that vanish in the limit $y^2 \rightarrow \infty$ (in the case $\Lambda > 0$) or in the limit $y^2 \rightarrow -1/\Lambda$ (in the case $\Lambda < 0$). Then we obtain

$$\left[(1 + \Lambda y^2) \frac{d^2}{dy^2} + \Lambda y \frac{d}{dy} - (1 + \Lambda) \left(\frac{y^2}{1 + \Lambda y^2} \right) \right] \Psi_\infty = -\Psi_\infty.$$

Thus, Ψ_∞ is the exact solution in the very particular case $e = 1/2$ and represents the asymptotic behaviour of the solution in the general case. Consequently, this property suggest the following factorization

$$\Psi(y, \Lambda) = h(y, \Lambda) (1 + \Lambda y^2)^{-1/(2\Lambda)}, \quad (9)$$

and then the new function $h(y, \Lambda)$ must satisfy the differential equation

$$(1 + \Lambda y^2)h'' + (\Lambda - 2)yh' + (2e - 1)h = 0, \quad h = h(y, \Lambda). \quad (10)$$

Let us first remark that, as $\lim_{\Lambda \rightarrow 0}(1 + \Lambda y^2) = 1$ and $\lim_{\Lambda \rightarrow 0}[1/(2\Lambda)] = \infty$, we have

$$\lim_{\Lambda \rightarrow 0}(1 + \Lambda y^2)^{1/(2\Lambda)} = \exp \left[\lim_{\Lambda \rightarrow 0} \frac{\log(1 + \Lambda y^2)}{2\Lambda} \right] = \exp \left[\lim_{\Lambda \rightarrow 0} \frac{y^2/(1 + \Lambda y^2)}{2} \right] = e^{y^2/2}.$$

Consequently,

$$\lim_{\Lambda \rightarrow 0} \Psi(y, \Lambda) = h(y, 0) e^{-(1/2)y^2}.$$

Furthermore, if we put $\Lambda = 0$ in the equation (8) we see that $\bar{h}(y) = h(y, 0)$ satisfies the Hermite equation

$$\Lambda \rightarrow 0 \quad \bar{h}'' - 2y\bar{h}' + (2e - 1)\bar{h} = 0$$

In other words, for $\Lambda = 0$, the function $h(y, 0)$ is a solution of Hermite equation. Recall that when $2e - 1$ is an even integer number, i.e. $2e - 1 = 2p$, such differential equation admits polynomial solutions, called (with an appropriately chosen overall factor) Hermite polynomials.

Nevertheless, it is also interesting to note that the dependence of the parameter Λ makes of (10) an equation with certain similarities with the Legendre's equation.

It is clear that the three coefficient functions of the linear differential equation

$$(1 + \Lambda y^2)h'' + (\Lambda - 2)yh' + (2e - 1)h = 0, \quad h = h(y, \Lambda),$$

are analytic at the origin (and the director coefficient does not vanish in a neighbourhood of $y = 0$). The origin is therefore an ordinary point and we expect an analytic solution h with a power series expansion convergent in an interval $(-R, R)$ for some non-zero value of the radius of convergence R ,

$$h(y, \Lambda) = \sum_{n=0}^{\infty} a_n(\Lambda) y^n = a_0(\Lambda) + a_1(\Lambda) y + a_2(\Lambda) y^2 + \dots$$

In this way the equation becomes

$$\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + \Lambda a_n y^n + (\Lambda - 2) n a_n + (2e - 1) a_n] y^n = 0, \quad (11)$$

and therefore it determines to the following Λ -dependent recursion relation

$$a_{n+2} = (-1) \frac{a_n}{(n+2)(n+1)} \left[n(\Lambda n - 2) + (2e - 1) \right], \quad n = 0, 1, 2, \dots$$

Note that this relation shows that, as in the $\Lambda = 0$ case, even power coefficients are related among themselves and the same is true for odd power coefficients. In both cases, having in mind that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n(\Lambda n - 2) + (2e - 1)}{(n+2)(n+1)} \right| |x^2| = |\Lambda| |x^2|$$

so that the radius of convergence R is therefore given by

$$R = \frac{1}{\sqrt{|\Lambda|}}.$$

Hence, when we consider the limit $\Lambda \rightarrow 0$, we recover the radius $R = \infty$ of the Hermite's equation.

The general solution h is given by the linear combination $h = a_0 h_1 + a_1 h_2$ where $h_1(y)$ and $h_2(y)$ are the solutions determined by $h_1(0) = 1$, $h'_1(0) = 0$ and $h_2(0) = 0$, $h'_2(0) = 1$, respectively.

The condition for the differential equation to admit a polynomial solution is the existence of an integer number p such that

$$2e - 1 = 2p - \Lambda p^2,$$

because then

$$a_p \neq 0, \quad a_{p+2} = 0,$$

and the solution is a polynomial of order p . This relation establishes some possible values for the spectrum of the Hamiltonian \hat{H} and therefore of H (this question will be discussed in the next subsection).

The polynomial solutions are given by

- Even index (even power polynomials)

$$\begin{aligned} \mathcal{H}_{2p} &= k_{2p} \sum_{r=0}^{r=p} a_{2r} y^{2r} \\ a_{2r} &= (-1)^r \frac{a_0}{2^r r!} p' (p' - 2)(p' - 4) \dots (p' - 2(r - 1)) \\ &\quad [2 - \Lambda p'] [2 - \Lambda(p' + 2)] [2 - \Lambda(p' + 4)] \dots [2 - \Lambda(p' + 2(r - 1))] \end{aligned}$$

where we have introduced the notation $p' = 2p$. More specifically, the expressions of the first solution h_1 , in the particular cases of $p' = 0, 2, 4, 6$, are given by:

$$\begin{aligned} \mathcal{H}_0 &= k_0 \\ \mathcal{H}_2 &= k_2 [1 - 2(1 - \Lambda)y^2] \\ \mathcal{H}_4 &= k_4 [1 - 4(1 - 2\Lambda)y^2 + (\frac{4}{3})(1 - 2\Lambda)(1 - 3\Lambda)y^4] \\ \mathcal{H}_6 &= k_6 [1 - 6(1 - 3\Lambda)y^2 + 4(1 - 3\Lambda)(1 - 4\Lambda)y^4 + (\frac{8}{15})(1 - 3\Lambda)(1 - 4\Lambda)(1 - 5\Lambda)y^6] \end{aligned}$$

where k_0, k_1, k_2, \dots , are overall multiplicative constants.

- Odd index (odd power polynomials)

$$\begin{aligned}\mathcal{H}_{2p+1} &= k_{2p+1} \sum_{r=0}^{r=p} a_{2r+1} y^{2r+1} \\ a_{2r+1} &= (-1)^{r+1} \frac{a_1}{2^{r+1} + 1!} (p' - 1)(p' - 3) \dots (p' - (2r - 1)) \\ &\quad [2 - \Lambda(p' + 1)][2 - \Lambda(p' + 3)] \dots [2 - \Lambda(p' + (2r - 1))]\end{aligned}$$

where we have introduced the notation $p' = 2p + 1$. More specifically, the expressions of the second solution h_2 for $p' = 1, 3, 5$, are given by:

$$\begin{aligned}\mathcal{H}_1 &= k_1 y \\ \mathcal{H}_3 &= k_3 \left[y - \left(\frac{2}{3}\right)(1 - 2\Lambda)y^3 \right] \\ \mathcal{H}_5 &= k_5 \left[y - \left(\frac{4}{3}\right)(1 - 3\Lambda)y^3 + \left(\frac{4}{15}\right)(1 - 3\Lambda)(1 - 4\Lambda)y^5 \right]\end{aligned}$$

where k_1, k_3, k_5, \dots , are arbitrary multiplicative constants.

It is clear that these particular polynomials solutions \mathcal{H}_m , $m = 0, 1, 2, \dots$, play, for this Λ -dependent oscillator, a similar role to the Hermite's polynomials for the standard harmonic oscillator.

2.2 λ -dependent Sturm-Liouville problem and orthogonality

The Λ -dependent differential equation

$$a_0 h'' + a_1 h' + a_2 h = 0,$$

$$a_0 = (1 + \Lambda y^2), \quad a_1 = (\Lambda - 2)y, \quad a_2 = (2e - 1),$$

is not self-adjoint since $a'_0 \neq a_1$ but it can be reduced to self-adjoint form by making use of the following integrating factor

$$\mu(y) = \left(\frac{1}{a_0}\right) e^{\int (a_1/a_0) dy} = (1 + \Lambda y^2)^{-(\Lambda+2)/(2\Lambda)}$$

in such a way that we arrive to the following expression

$$\frac{d}{dy} \left[p(y, \Lambda) \frac{dh}{dy} \right] + (2e - 1) r(y, \Lambda) h = 0,$$

where the two functions $p = p(y, \Lambda)$ and $r = r(y, \Lambda)$ are given by

$$\begin{aligned}p(y, \Lambda) &= e^{\int (a_1/a_0) dy} = (1 + \Lambda y^2)^{1/2-1/\Lambda}, \\ r(y, \Lambda) &= \left(\frac{a_2}{a_0}\right) e^{\int (a_1/a_0) dx} = \frac{1}{1 + \Lambda y^2} (1 + \Lambda y^2)^{1/2-1/\Lambda}.\end{aligned}$$

Thus, we have obtained

$$\frac{d}{dy} \left[\left(\frac{\sqrt{1 + \Lambda y^2}}{(1 + \Lambda y^2)^{1/\Lambda}} \right) \frac{dh}{dy} \right] + \frac{(2e - 1) h}{\sqrt{1 + \Lambda y^2} (1 + \Lambda y^2)^{1/\Lambda}} = 0 \quad (12)$$

that, together with appropriate conditions for the behaviour of the solutions at the end points, constitute a Sturm-Liouville problem. It is to be pointed out that the boundary conditions are in fact different according to the sign of Λ ; therefore we arrive to, not just one, but two different Sturm-Liouville problems:

- If Λ is negative the range of the variable y is limited by the restriction $y^2 < 1/|\Lambda|$. In this case the problem, defined in the bounded interval $[-a_\Lambda, a_\Lambda]$ with $a_\Lambda = 1/\sqrt{|\Lambda|}$, is singular because the function $p(y, \Lambda)$ vanishes in the two end points $y_1 = -a_\Lambda$ and $y_2 = a_\Lambda$. The conditions to be imposed in this case lead to prescribe that the solutions $h(y, \Lambda)$ of the problem must be bounded functions at the two end points, $y_1 = -a_\Lambda$ and $y_2 = a_\Lambda$, of the interval. It is clear that this leads to the above mentioned polynomial solutions.
- If Λ is positive the variable y is defined in the whole real line \mathbb{R} and, therefore, the Sturm-Liouville problem is singular. The solutions $h(y, \Lambda)$ must be well defined in all \mathbb{R} , and the boundary conditions prescribe that the behaviour of these functions when $y \rightarrow \pm \infty$ must be such that their norms, determined with respect to the weight function $r(y)$, be finite. It is clear that in this case the solutions of the problem are again the Λ -dependent polynomials \mathcal{H}_m , $m = 0, 1, 2, \dots$

Proposition 1 *The eigenfunctions of the problem (12) are orthogonal with respect to the function $r = (1 + \Lambda y^2)^{-(1/2+1/\Lambda)}$.*

Proof: This statement is just a consequence of the properties of the Sturm-Liouville problems.

Because of this the polynomial solutions \mathcal{H}_m , $m = 0, 1, 2, \dots$, of the equation (10), satisfy

$$\int_{-a_\Lambda}^{a_\Lambda} \frac{\mathcal{H}_m(y, \Lambda) \mathcal{H}_n(y, \Lambda)}{(1 + \Lambda y^2)^{1/\Lambda} \sqrt{1 + \Lambda y^2}} dy = 0, \quad m \neq n, \quad \Lambda < 0, \quad (13)$$

and

$$\int_{-\infty}^{\infty} \frac{\mathcal{H}_m(y, \Lambda) \mathcal{H}_n(y, \Lambda)}{(1 + \Lambda y^2)^{1/\Lambda} \sqrt{1 + \Lambda y^2}} dy = 0, \quad m \neq n, \quad \Lambda > 0. \quad (14)$$

If we define the Λ -dependent Hermite functions Ψ_m by

$$\Psi_m(y, \Lambda) = \mathcal{H}_m(y, \Lambda) (1 + \Lambda y^2)^{-1/(2\Lambda)}, \quad m = 0, 1, 2, \dots$$

then the above statement admits the following alternative form: *The Λ -dependent Hermite functions $\Psi_m(y, \Lambda)$ are orthogonal with respect to the weight function $\tilde{r} = 1/\sqrt{1 + \Lambda y^2}$:*

$$\int_{-a_\Lambda}^{a_\Lambda} \Psi_m(y, \Lambda) \Psi_n(y, \Lambda) \tilde{r}(y, \Lambda) dy = \int_{-a_\Lambda}^{a_\Lambda} \Psi_m(y, \Lambda) \Psi_n(y, \Lambda) \frac{dy}{\sqrt{1 + \Lambda y^2}} = 0, \quad m \neq n, \quad \Lambda < 0, \quad (15)$$

and

$$\int_{-\infty}^{\infty} \Psi_m(y, \Lambda) \Psi_n(y, \Lambda) \tilde{r}(y, \Lambda) dy = \int_{-\infty}^{\infty} \Psi_m(y, \Lambda) \Psi_n(y, \Lambda) \frac{dy}{\sqrt{1 + \Lambda y^2}} = 0, \quad m \neq n, \quad \Lambda > 0. \quad (16)$$

Note that this orthogonality with respect to the weight function \tilde{r} coincides with the orthogonality with respect to the measure $d\mu_\Lambda$ discussed in the subsection 2.1.

Finally, we also remark that

$$\lim_{\Lambda \rightarrow 0} \Psi_m(y, \Lambda) = H_m(y) e^{-(1/2)y^2}, \quad m = 0, 1, 2, \dots$$

2.3 Wave functions and energy levels

We started by introducing the factorization (9) for the wave functions Ψ and then, when the boundary conditions have been taken into account, we have obtained that the first factor $h(y, \Lambda)$ must be a polynomial function of a very particular class. The result is that the wave functions become “a Λ -dependent polynomial divided by the function $(1 + \Lambda y^2)^{1/(2\Lambda)}$ ”; more specifically,

$$\Psi_m(y, \Lambda) = \mathcal{H}_m(y, \Lambda) (1 + \Lambda y^2)^{-1/(2\Lambda)}, \quad m = 0, 1, 2, \dots$$

where m is the order of the polynomial. Since the second factor is even and has no nodes, the corresponding eigenfunction $\Psi_m(y, \Lambda)$ has the same parity as m and m nodes (when $\Lambda < 0$ the nodes are inside the bounded interval $[-a_\Lambda, a_\Lambda]$). Therefore, these two particular properties, parity and number of nodes, are preserved by the deformation introduced by Λ .

The two cases Λ negative and Λ positive are rather different and must be studied separately.

(i) Let us suppose that Λ is negative, $\Lambda < 0$.

In this case the dynamics is restricted to the interval $y^2 < 1/|\Lambda|$ and the boundary conditions imply that the function $\Psi(y, \Lambda)$ must vanish at the end points $y_1 = -1/\sqrt{|\Lambda|}$ and $y_2 = 1/\sqrt{|\Lambda|}$. This means that the first factor $h(y, \Lambda)$ must be a polynomial and, because of this, the eigenfunctions are given by a countable family of functions Ψ_m , $m = 0, 1, 2, \dots$. The eigenfunctions Ψ_m and the eigenvalues e_m are given by

$$\begin{aligned} \Psi_m(y, \Lambda) &= \mathcal{H}_m(y, \Lambda) (1 - |\Lambda| y^2)^{1/(2|\Lambda|)}, \\ e_m &= \left(m + \frac{1}{2}\right) + \frac{1}{2} m^2 |\Lambda|, \quad m = 0, 1, 2, \dots, m, \dots \end{aligned} \quad (17)$$

These eigenfunctions Ψ_m are well defined and vanish at the end points, y_1 and y_2 , for all the integers values of m without any restriction; thus, there exists a infinite but countable number of eigenfunctions. The energy spectrum is unbounded but, contrarily to what happens in the case of the linear oscillator, is not equispaced

$$e_0 < e_1 < e_2 < e_3 < \dots < e_m < e_{m+1} < \dots$$

$$e_{m+1} - e_m = 1 + \left(m + \frac{1}{2}\right) |\Lambda|.$$

Nevertheless, it is also true that the energy e_0 of the ground-state Ψ_0 is $e_0 = 1/2$; that is, the zero-point energy is given by $E_0 = (1/2)\hbar \alpha$ for any $\Lambda \neq 0$, as in the linear case.

(ii) Let us suppose that Λ is positive, $\Lambda > 0$.

This case is rather delicate and deserves to be analyzed with more detail. It is convenient to recall that the domain of y is now the whole real line \mathbb{R} and hence it is necessary to take into account the problem of the convergence at the infinity. In fact, it is necessary that the following integral be convergent

$$\int_{-\infty}^{\infty} \frac{\mathcal{H}_m^2(y, \Lambda)}{(1 + \Lambda y^2)^{1/\Lambda} \sqrt{1 + \Lambda y^2}} dy < \infty$$

and, as for large values of y , the powers of the dominant terms in the numerator and the denominator are $2m$ and $1 + 2/\Lambda$ respectively, we arrive to a certain condition to be satisfied by m . In fact,

given a certain value of Λ , then the admissible functions Ψ_m are those associated to integer values of m satisfying the condition

$$m < m_\Lambda = \frac{1}{\Lambda}.$$

This means that for large values of Λ the oscillator only admits the fundamental level Ψ_0 as eigenfunction, while for smaller values of Λ the oscillator admits a certain finite number of eigenstates, in such a way that when Λ decreases this number increases and in the linear limit $\Lambda \rightarrow 0$ the number goes to infinity. The eigenfunctions Ψ_m and the eigenvalues e_m are given by

$$\begin{aligned} \Psi_m(y, \Lambda) &= \mathcal{H}_m(y, \Lambda) (1 + \Lambda y^2)^{-1/(2\Lambda)}, \\ e_m &= \left(m + \frac{1}{2}\right) - \frac{1}{2} m^2 \Lambda, \quad m = 0, 1, 2, \dots, N_\Lambda, \end{aligned} \quad (18)$$

where N_Λ denotes the greatest integer lower than m_Λ . The energy spectrum is bounded and again not equispaced

$$\begin{aligned} e_0 &< e_1 < e_2 < e_3 < \dots < e_m < e_{N_\Lambda}, \\ e_{m+1} - e_m &= 1 - \left(m + \frac{1}{2}\right) \Lambda. \end{aligned}$$

It can be seen that the energy e_m , considered as a function of m , is an increasing function for low values of m and has a maximum at the point $1/\Lambda$. Beyond that point it becomes a decreasing function and it even takes negative values; but, due to the above condition $m < m_\Lambda$ for physical eigenfunctions, this behaviour of e_m for large values of m is not physically meaningful. Figure III shows the form of e_m as a function of m for two particular values of Λ .

The following particular cases, that are presented in a decreasing way, illustrate the situation for several positive values of Λ

- If $\Lambda \geq 1$ the only bound state of this deformed oscillator is the fundamental state Ψ_0 .
- If $1 > \Lambda \geq 1/2$ there are two eigenfunctions; the fundamental level Ψ_0 and another bound state Ψ_1 with energies $e_0 < e_1$. For example, for $\Lambda = 0.8$ we have $e_0 = 1/2$ and $e_1 = 1.1$.
- If $1/2 > \Lambda \geq 1/3$ there are three eigenfunctions Ψ_0, Ψ_1, Ψ_2 , with energies $e_0 < e_1 < e_2$. For example, for $\Lambda = 0.4$ we have $e_0 = 1/2$, $e_1 = 1.3$, and $e_2 = 1.7$.
- If $1/3 > \Lambda \geq 1/4$ there are four eigenfunctions $\Psi_0, \Psi_1, \Psi_2, \Psi_3$, with energies $e_0 < e_1 < e_2 < e_3$. For example, for $\Lambda = 0.3$ we have $e_0 = 1/2$, $e_1 = 1.35$, $e_2 = 1.90$, and $e_3 = 2.15$.
- If $1/(n-1) > \Lambda \geq 1/n$ there exist n eigenfunctions (the fundamental one and $n-1$ other excited bound states) Ψ_0 and Ψ_i , $i = 1, 2, \dots, n-1$, with energies $e_0 = 1/2 < e_1 < e_2 < e_3 < \dots < e_{n-1}$.

We close this section with a comparison of the energy levels $e_m(\Lambda)$ of the Λ -dependent oscillator with the corresponding values of the linear harmonic oscillator

$$\left. \begin{aligned} e_m(\Lambda) &= e_m(0) + \frac{1}{2} m^2 |\Lambda|, & \Lambda < 0 \\ e_m(\Lambda) &= e_m(0) - \frac{1}{2} m^2 \Lambda, & \Lambda > 0 \end{aligned} \right\} \quad (19)$$

It is clear, therefore, that when Λ is negative the energy $e_m(\Lambda)$ is higher than the energy $e_m(0)$ of the harmonic oscillator and when $\Lambda > 0$, $e_m(\Lambda)$, $m < m_\Lambda$, is lower than $e_m(0)$. Figure IV shows this property by plotting the values of $e_m(\Lambda)$ for $\Lambda = -0.30$ and $\Lambda = 0.30$.

3 Schrödinger factorization formalism

The factorization of Schrödinger Hamiltonians in terms of first order differential operators [41]-[45] has been shown to be very efficient in finding properties of the spectrum of the Hamiltonian and even exhaustively solving the spectral problem in the case of shape invariant Hamiltonians [46]-[53].

In the usual case one starts with a Hamiltonian H_1 like

$$H_1 = -\frac{d^2}{dx^2} + V_1(x),$$

determined by the potential V_1 and looks for two operators a and a^+ of the form

$$a = \frac{d}{dx} + W(x), \quad a^+ = -\frac{d}{dx} + W(x),$$

such that H_1 admits the factorization

$$H_1 = a^+ a = \left[-\frac{d}{dx} + W(x) \right] \left[\frac{d}{dx} + W(x) \right]. \quad (20)$$

This happens if and only if the function W is such that

$$W' - W^2 + V_1 = 0,$$

i.e., W is a solution of the preceding Riccati equation. When H_1 admits such a factorization we can define a new Hamiltonian H_2 , of the form

$$H_2 = -\frac{d^2}{dx^2} + V_2(x),$$

by means of the new factorization

$$H_2 = a a^+ = \left[\frac{d}{dx} + W(x) \right] \left[-\frac{d}{dx} + W(x) \right]. \quad (21)$$

Such a Hamiltonian H_2 is said to be the partner of H_1 , and leads to a new Riccati equation

$$W' + W^2 - V_2 = 0.$$

In summary, the function W is a solution of two different Riccati equations, determined by the two potentials V_1 and V_2 :

$$\begin{aligned} W' - W^2 + V_1 &= 0, \\ W' + W^2 - V_2 &= 0, \end{aligned}$$

Therefore, the two potentials V_1 and V_2 are related by

$$V_2 = 2W^2 - V_1 .$$

The important point is that a is an intertwining operator for both Hamiltonians and this property allows to establish relations between the corresponding spectral problems.

The method can be adapted to deal with the class of Hamiltonians we are interested in:

$$\hat{H}_1 = -(1 + \lambda x^2) \frac{d^2}{dx^2} - \lambda x \frac{d}{dx} + U_1(x) ,$$

which correspond to a problem with a position dependent mass; this is a particular example or factorization of Sturm–Liouville operators [59]. In this case we should look for two operators A and A^+ of the following form

$$\begin{aligned} A &= \sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) , \\ A^+ &= -\sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) , \end{aligned}$$

and then, as

$$\hat{H}_1 = A^+ A = \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) \right] \left[\sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) \right] \quad (22)$$

such a factorization is possible if and only if the function W satisfies the Riccati differential equation

$$\sqrt{1 + \lambda x^2} W' - W^2 + U_1 = 0 .$$

The partner Hamiltonian is now of the form

$$\hat{H}_2 = -(1 + \lambda x^2) \frac{d^2}{dx^2} - \lambda x \frac{d}{dx} + U_2(x) ,$$

and is defined by the alternative factorization

$$\hat{H}_2 = A A^+ = \left[\sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) \right] \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + W(x) \right] \quad (23)$$

which leads to

$$\sqrt{1 + \lambda x^2} W' + W^2 - U_2 = 0 .$$

In other words, the function W defining the factorization should be a solution of two Riccati equations defined by the two potential functions U_1 and U_2

$$\begin{aligned} \sqrt{1 + \lambda x^2} W' - W^2 + U_1 &= 0 , \\ \sqrt{1 + \lambda x^2} W' + W^2 - U_2 &= 0 , \end{aligned}$$

and therefore both potentials are related by

$$U_2 = 2W^2 - U_1 .$$

4 Schrödinger factorization approach to the λ -dependent nonlinear oscillator

In a recent paper [40] a factorization for the λ -dependent quantum nonlinear oscillator was proposed and it was shown to have the shape-invariance property, therefore the spectrum of such system has been fully determined. We aim in this section to do a more complete computation of both the spectrum and the corresponding eigenfunctions and to study some interesting properties of the polynomial replacing Hermite polynomials in this deformed case.

4.1 Operators A and A^+ and intertwined Hamiltonians H_1 and H_2

It has been proved in [40] that the following operators A and A^+

$$\begin{aligned} A &= \frac{\hbar}{\sqrt{2m}} \left[\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) \frac{x}{\sqrt{1 + \lambda x^2}} \right], \\ A^+ &= \frac{\hbar}{\sqrt{2m}} \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) \frac{x}{\sqrt{1 + \lambda x^2}} \right], \end{aligned}$$

provide a factorization for the λ -dependent quantum nonlinear oscillator. In fact, if we denote by W_λ the following function

$$W_\lambda = \frac{x}{\sqrt{1 + \lambda x^2}},$$

then the product $A^+ A$ that is given by

$$A^+ A = \frac{\hbar^2}{2m} \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) W_\lambda \right] \left[\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) W_\lambda \right],$$

turns out to be

$$\begin{aligned} A^+ A &= -\frac{\hbar^2}{2m} \left[(1 + \lambda x^2) \frac{d^2}{dx^2} + \lambda x \frac{d}{dx} \right] + U_1, \\ U_1 &= \frac{1}{2} (m\alpha) \left(\alpha + \frac{\hbar}{m} \lambda \right) W_\lambda^2 - \frac{1}{2} (\hbar\alpha). \end{aligned} \tag{24}$$

In a similar way, the product $A A^+$ given by

$$A A^+ = \frac{\hbar^2}{2m} \left[\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) W_\lambda \right] \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + \left(\frac{m\alpha}{\hbar} \right) W_\lambda \right],$$

turns out to be

$$\begin{aligned} A A^+ &= -\frac{\hbar^2}{2m} \left[(1 + \lambda x^2) \frac{d^2}{dx^2} + \lambda x \frac{d}{dx} \right] + U_2, \\ U_2 &= \frac{1}{2} (m\alpha) \left(\alpha - \frac{\hbar}{m} \lambda \right) W_\lambda^2 + \frac{1}{2} (\hbar\alpha). \end{aligned} \tag{25}$$

Consequently, the Hamiltonian \hat{H} given by (6), is such that

$$\hat{H} = \hat{H}_1 + \frac{1}{2} (\hbar\alpha) = \hat{H}_2 - \frac{1}{2} (\hbar\alpha)$$

where \hat{H}_1 and \hat{H}_2 are the Hamiltonian $\hat{H}_1 = A^\dagger A$ and its partner $\hat{H}_2 = A A^\dagger$.

Note that the limits when $\lambda \rightarrow 0$ are given by:

$$\begin{aligned}\lim_{\lambda \rightarrow 0} \hat{H}_1 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \left(\frac{1}{2}\right) m \alpha^2 x^2 - \left(\frac{1}{2}\right) (\hbar \alpha), \\ \lim_{\lambda \rightarrow 0} \hat{H}_2 &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \left(\frac{1}{2}\right) m \alpha^2 x^2 + \left(\frac{1}{2}\right) (\hbar \alpha).\end{aligned}$$

and note also that the commutator $[A, A^\dagger]$ is not a constant. In fact,

$$[A, A^\dagger] = \left(1 - \frac{\lambda x^2}{1 + \lambda x^2}\right) (\hbar \alpha)$$

but when $\lambda \rightarrow 0$ we find

$$\lim_{\lambda \rightarrow 0} [A, A^\dagger] = \hbar \alpha.$$

4.2 The spectrum of the quantum deformed nonlinear oscillator

The two operators, the Hamiltonian \hat{H}_1 and its partner \hat{H}_2 , depend on the parameter α ; the important point is that this α -dependence determines an important relation between them given by

$$\hat{H}_2(\alpha) = \hat{H}_1(\alpha_1) + R(\alpha_1), \quad (26)$$

where α_1 denotes $\alpha_1 = f(\alpha)$ with the functions f and R defined by the expressions

$$f(\alpha) = \alpha - \frac{\hbar}{m} \lambda, \quad R(\alpha) = \hbar \alpha + \left(\frac{1}{2}\right) \frac{\hbar^2}{m} \lambda. \quad (27)$$

The condition (26) is called ‘shape invariance’ condition and for such cases there exists a recipe to compute the spectrum and to determine the corresponding eigenfunctions [47, 48] (see also [46] for a modern approach).

More specifically, as a first step, the bound state $|\Psi_0(\alpha)\rangle$ is found by solving $A(\alpha)|\Psi_0(\alpha)\rangle = 0$, and has a zero energy, i.e. $\hat{H}_1(\alpha)|\Psi_0(\alpha)\rangle = 0$. Then, using (26) we can see that $|\Psi_0(\alpha_1)\rangle$ is an eigenstate of $\hat{H}_2(\alpha)$ with an energy $E_1 = R(\alpha_1)$, because

$$\hat{H}_2(\alpha)|\Psi_0(\alpha_1)\rangle = (\hat{H}_1(\alpha_1) + R(\alpha_1))|\Psi_0(\alpha_1)\rangle = R(\alpha_1)|\Psi_0(\alpha_1)\rangle. \quad (28)$$

Next, $A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle$ is an eigenstate of $\hat{H}_1(\alpha)$ with an energy $E_1 = R(\alpha_1)$, because if we use the intertwining relation $\hat{H}_1(\alpha)A^\dagger(\alpha) = A^\dagger(\alpha)\hat{H}_2(\alpha)$ we see that

$$\hat{H}_1(\alpha)A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle = A^\dagger(\alpha)\hat{H}_2(\alpha)|\Psi_0(\alpha_1)\rangle = A^\dagger(\alpha)(\hat{H}_1(\alpha_1) + R(\alpha_1))|\Psi_0(\alpha_1)\rangle,$$

and hence we arrive to

$$\hat{H}_1(\alpha)A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle = R(\alpha_1)A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle.$$

This process should be iterated and we will find the sequence of energies for $\hat{H}_1(\alpha)$

$$E_k = \sum_{j=1}^k R(\alpha_j), \quad E_0 = 0, \quad (29)$$

the corresponding eigenfunctions being

$$|\Psi_n(\alpha_0)\rangle = A^\dagger(\alpha_0)A^\dagger(\alpha_1)\cdots A^\dagger(\alpha_{n-1})|\Psi_0(\alpha_n)\rangle, \quad (30)$$

where $\alpha_0 = \alpha$ and $\alpha_j = f(\alpha_{j-1})$, namely, $\alpha_j = f^j(\alpha_0) = f^j(\alpha)$.

We can now apply this procedure with the functions f and R given by (27). In this case we obtain

$$\begin{aligned} \alpha_k &= f^k(\alpha) = \alpha - \frac{\hbar\lambda}{m}k, \\ R(\alpha_k) &= \hbar\alpha_k + \frac{\hbar^2}{2m}\lambda = \hbar\left(\alpha - \frac{\hbar\lambda}{m}k\right) + \frac{\hbar^2}{2m}\lambda \end{aligned} \quad (31)$$

and therefore the energy E_n that is given by

$$E_n = \sum_{k=1}^n R(\alpha_k) = \sum_{k=1}^n \left(\hbar\left(\alpha - \frac{\hbar\lambda}{m}k\right) + \frac{\hbar^2}{2m}\lambda \right)$$

becomes

$$E_n = \left(n\alpha - \frac{\hbar\lambda}{2m}n(n+1) \right) \hbar + \frac{n\hbar^2}{2m}\lambda = \left(n\alpha - \frac{n^2\hbar\lambda}{2m} \right) \hbar,$$

that in terms of the adimensional variable Λ becomes

$$E_n = \left[n - \left(\frac{1}{2}\right)n^2\Lambda \right] (\hbar\alpha), \quad n = 0, 1, 2, \dots \quad (32)$$

The eigenstates of \hat{H} will include an additional term $(1/2)\hbar\alpha$ which shifts all levels.

4.3 Eigenfunctions of the quantum deformed nonlinear oscillator

As indicated before, the fundamental state is determined by $A|\Psi_0\rangle = 0$. The wave function $\Psi_0(x)$ in the coordinate representation is then given by the normalized solution of the linear first order equation:

$$(1 + \lambda x^2) \frac{d}{dx} \Psi_0 + \beta x \Psi_0 = 0, \quad \beta = \frac{m\alpha}{\hbar},$$

whose solution is proportional to

$$\Psi_0 = \frac{1}{(1 + \lambda x^2)^{r_0}}, \quad r_0 = \frac{\beta}{2\lambda} = \frac{1}{2\Lambda}.$$

In the limit of λ going to zero, having in mind that $\lim_{\lambda \rightarrow 0}(1 + \lambda x^2) = 1$ and $\lim_{\lambda \rightarrow 0}\beta/(2\lambda) = \infty$, we have that

$$\lim_{\lambda \rightarrow 0}(1 + \lambda x^2)^{-\beta/2\lambda} = \exp\left(-\lim_{\lambda \rightarrow 0} \frac{\log(1 + \lambda x^2)}{2\lambda/\beta}\right) = \exp\left(-\frac{\beta}{2}x^2\right).$$

The other eigenfunctions are obtained according to the recipe we derived before as follows:

$$\begin{aligned}
\Psi_1 &= A^+(\alpha_0) \Psi_0(\alpha_1) \\
\Psi_2 &= A^+(\alpha_0) A^+(\alpha_1) \Psi_0(\alpha_2) \\
\Psi_3 &= A^+(\alpha_0) A^+(\alpha_1) A^+(\alpha_2) \Psi_0(\alpha_3) \\
&\dots \\
\Psi_n &= A^+(\alpha_0) A^+(\alpha_1) A^+(\alpha_2) \dots A^+(\alpha_{n-1}) \Psi_0(\alpha_n)
\end{aligned} \tag{33}$$

In the next property we establish a powerful tool for computing the action of different operators $A^+(\alpha_k)$ on functions $\Psi_0(\alpha_j)$.

Proposition 2 *Let $g(x)$ be an arbitrary differentiable function. Then the following equality holds:*

$$(1 + \lambda x^2)^p \sqrt{1 + \lambda x^2} \frac{d}{dx} (1 + \lambda x^2)^{-p} g(x) = (-1) \left[-\sqrt{1 + \lambda x^2} \frac{d}{dx} + \frac{(2p\lambda)x}{\sqrt{1 + \lambda x^2}} \right] g(x).$$

Proof: This property is proved by direct computation.

Using the notation $z_x = 1 + \lambda x^2$, the above property can be rewritten as follows

$$z_x^p \sqrt{z_x} \frac{d}{dx} z_x^{-p} = (-1) \left[-\sqrt{z_x} \frac{d}{dx} + \frac{(2p\lambda)x}{\sqrt{z_x}} \right],$$

in such a way that in the particular case $p = b/2\lambda$ it reduces to

$$z_x^{b/2\lambda} \sqrt{z_x} \frac{d}{dx} z_x^{-b/2\lambda} = (-1) \left[-\sqrt{z_x} \frac{d}{dx} + \frac{bx}{\sqrt{z_x}} \right].$$

The operators $A^+(\alpha_0)$, $A^+(\alpha_1)$, $A^+(\alpha_2)$, \dots , which are given by

$$\begin{aligned}
A^+(\alpha_0) &= C_0 \left[-\sqrt{z_x} \frac{d}{dx} + \left(\frac{m\alpha_0}{\hbar} \right) \frac{x}{\sqrt{z_x}} \right], \\
A^+(\alpha_1) &= C_1 \left[-\sqrt{z_x} \frac{d}{dx} + \left(\frac{m\alpha_1}{\hbar} \right) \frac{x}{\sqrt{z_x}} \right], \\
A^+(\alpha_2) &= C_2 \left[-\sqrt{z_x} \frac{d}{dx} + \left(\frac{m\alpha_2}{\hbar} \right) \frac{x}{\sqrt{z_x}} \right], \\
&\dots \dots \dots
\end{aligned}$$

can be rewritten as follows

$$\begin{aligned}
A^+(\alpha_0) &= C_0 (-1) z_x^{\beta_0/2\lambda} \sqrt{z_x} \frac{d}{dx} z_x^{-\beta_0/2\lambda}, \\
A^+(\alpha_1) &= C_1 (-1) z_x^{\beta_1/2\lambda} \sqrt{z_x} \frac{d}{dx} z_x^{-\beta_1/2\lambda}, \\
A^+(\alpha_2) &= C_2 (-1) z_x^{\beta_2/2\lambda} \sqrt{z_x} \frac{d}{dx} z_x^{-\beta_2/2\lambda}, \\
&\dots \dots \dots
\end{aligned}$$

where β_0 and β_k denote $\beta_0 = \beta$ and $\beta_k = (m\alpha_k)/\hbar$.

Finally, the functions $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \dots, \Psi_n, \dots$, turn out to be

$$\begin{aligned}\Psi_0 &= C_0 z_x^{-\beta/2\lambda}, \\ \Psi_1 &= C_0 C_1 (-1) z_x^{-\beta/2\lambda} [z_x^{\beta/\lambda} \cdot \sqrt{z_x} \frac{d}{dx} (\sqrt{z_x}) z_x^{-\beta_0/\lambda}], \\ \Psi_2 &= (C_0 \dots C_2) (-1)^2 z_x^{-\beta/2\lambda} [z_x^{\beta/\lambda} \cdot \sqrt{z_x} \frac{d^2}{dx^2} (\sqrt{z_x})^3 z_x^{-\beta_0/\lambda}], \\ \Psi_3 &= (C_0 \dots C_3) (-1)^3 z_x^{-\beta/2\lambda} [z_x^{\beta/\lambda} \cdot \sqrt{z_x} \frac{d^3}{dx^3} (\sqrt{z_x})^5 z_x^{-\beta_0/\lambda}], \\ \dots &\dots\dots\dots\end{aligned}$$

and therefore the n -th wave-function $\Psi_n(x)$ is given by

$$\Psi_n = (C_0 \dots C_n) (-1)^n \left[z_x^{\beta/\lambda} \cdot \sqrt{z_x} \frac{d^n}{dx^n} (\sqrt{z_x})^{2n-1} z_x^{-\beta_0/\lambda} \right] z_x^{-\beta/2\lambda} \quad (34)$$

that must be considered as a new way of representing the expression (9) obtained in the Sec. II by the direct approach to the differential equation.

5 λ -dependent Hermite polynomials

We have solved the λ -dependent nonlinear oscillator by using two different procedures and in both cases we have arrived to an expression for the wave-function $\Psi_n(x)$ depending of some λ -dependent polynomials that must be considered as a deformation of the Hermite polynomials. A natural idea is to expect that the classical properties of the Hermite polynomials must remain true, with the appropriate modifications, for these new λ -dependent Hermite polynomials. In Sec. II we proved the orthogonality, now we will study three other fundamental properties: λ -dependent Rodrigues formula, the existence of a λ -dependent generating function and the existence of recursion relations among polynomials of different orders.

The Schrödinger factorization approach to the λ -dependent nonlinear oscillator, studied in Sec. IV, has provided the expression (34) for the wave function Ψ_n . Therefore, we have

$$\mathcal{H}_n(x, \lambda) = (-1)^n z_x^{\beta/\lambda+1/2} \frac{d^n}{dx^n} \left[z_x^n z_x^{-(\beta/\lambda+1/2)} \right], \quad n = 0, 1, 2, \dots$$

or

$$\mathcal{H}_n(y, \Lambda) = (-1)^n z_y^{1/\Lambda+1/2} \frac{d^n}{dy^n} \left[z_y^n z_y^{-(1/\Lambda+1/2)} \right], \quad z_y = 1 + \Lambda y^2, \quad (35)$$

that must be considered as the “Rodrigues formula” for the λ -dependent Hermite polynomials obtained in Sec. II. In fact, it is clear that

$$\lim_{\Lambda \rightarrow 0} \left[(-1)^n z_y^{1/\Lambda+1/2} \frac{d^n}{dy^n} \left[z_y^n z_y^{-(1/\Lambda+1/2)} \right] \right] = (-1)^n e^y \frac{d^n}{dy^n} e^{-y}$$

and consequently

$$\lim_{\Lambda \rightarrow 0} \mathcal{H}_n(y, \Lambda) = H_n(y), \quad n = 0, 1, 2, \dots$$

In a similar way to the standard Hermite polynomials, every function \mathcal{H}_m is a polynomial of degree m and it has a well defined parity

$$\mathcal{H}_m = c_m y^m + c_{m-2} y^{m-2} + c_{m-4} y^{m-4} + \dots$$

in such a way that the coefficient c_m of x^m in \mathcal{H}_m is given by

$$c_m = \prod_{r=m}^{2m-1} (2 - r\Lambda) = (2 - m\Lambda)(2 - (m+1)\Lambda) \dots (2 - (2m-1)\Lambda)$$

so that

$$\lim_{\Lambda \rightarrow 0} c_m = 2^m.$$

The first polynomials $\mathcal{H}_n = \mathcal{H}_n(y, \Lambda)$, $n = 0, 1, 2, \dots, 6$, have the following expressions

$$\begin{aligned} \mathcal{H}_0 &= 1 \\ \mathcal{H}_1 &= k_1 y \\ \mathcal{H}_2 &= k_2 [2(1 - \Lambda)y^2 - 1] \\ \mathcal{H}_3 &= k_3 [2(1 - 2\Lambda)y^3 - 3y] \\ \mathcal{H}_4 &= k_4 [4(1 - 2\Lambda)(1 - 3\Lambda)y^4 - 12(1 - 2\Lambda)y^2 + 3] \\ \mathcal{H}_5 &= k_5 [4(1 - 3\Lambda)(1 - 4\Lambda)y^5 - 20(1 - 3\Lambda)y^3 + 15y] \\ \mathcal{H}_6 &= k_6 [8(1 - 3\Lambda)(1 - 4\Lambda)(1 - 5\Lambda)y^6 - 60(1 - 3\Lambda)(1 - 4\Lambda)y^4 + 90(1 - 3\Lambda)y^2 - 15] \end{aligned}$$

where the constants k_i , $i = 1, 2, \dots, 6$, are given by

$$\begin{aligned} k_1 &= (2 - \Lambda), & k_2 &= (2 - 3\Lambda), \\ k_3 &= (2 - 3\Lambda)(2 - 5\Lambda), & k_4 &= (2 - 5\Lambda)(2 - 7\Lambda), \\ k_5 &= (2 - 5\Lambda)(2 - 7\Lambda)(2 - 9\Lambda), & k_6 &= (2 - 7\Lambda)(2 - 9\Lambda)(2 - 11\Lambda). \end{aligned}$$

We next analyze the existence of a generating function.

A generating function for the Λ -dependent Hermite polynomials $\mathcal{H}_n(y, \Lambda)$ must be a function $\mathcal{F}(t, y, \Lambda)$ that generates such polynomials as the successive coefficients of its Taylor power series of the additional variable t . Since the generating function of the Hermite polynomials is known to be given by

$$F(t, y) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) H_n(y) t^n, \quad F(t, y) = e^{(2ty - t^2)},$$

then, a necessary condition to be satisfied by $\mathcal{F}(t, y, \Lambda)$ must be the fulfillment of the limit

$$\lim_{\Lambda \rightarrow 0} \mathcal{F}(t, y, \Lambda) = e^{(2ty - t^2)}.$$

We propose to consider the function

$$\mathcal{F}(t, y, \Lambda) = \left(1 + \Lambda(2ty - t^2)\right)^{1/\Lambda} \quad (36)$$

as the appropriate function for representing the Λ -dependent generating function. It is clear that it satisfies the correct limit

$$\lim_{\Lambda \rightarrow 0} \left(1 + \Lambda(2ty - t^2)\right)^{1/\Lambda} = e^{(2ty - t^2)}.$$

The power series of \mathcal{F} in the new variable t is given by

$$\left(1 + \Lambda(2ty - t^2)\right)^{1/\Lambda} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) \tilde{\mathcal{H}}_n(y, \Lambda) t^n \quad (37)$$

where we have used the notation $\tilde{\mathcal{H}}_n$ for the coefficient of the Taylor series. The first polynomials $\tilde{\mathcal{H}}_n = \tilde{\mathcal{H}}_n(y, \Lambda)$, $n = 0, 1, 2, \dots, 6$, obtained in such a way, have the following expressions

$$\begin{aligned}\tilde{\mathcal{H}}_0 &= 1 \\ \tilde{\mathcal{H}}_1 &= 2g_1 y \\ \tilde{\mathcal{H}}_2 &= 2g_2 [2(1 - \Lambda)y^2 - 1] \\ \tilde{\mathcal{H}}_3 &= 4g_3 [2(1 - 2\Lambda)y^3 - 3y] \\ \tilde{\mathcal{H}}_4 &= 4g_4 [4(1 - 2\Lambda)(1 - 3\Lambda)y^4 - 12(1 - 2\Lambda)y^2 + 3] \\ \tilde{\mathcal{H}}_5 &= 8g_5 [4(1 - 3\Lambda)(1 - 4\Lambda)y^5 - 20(1 - 3\Lambda)y^3 + 15y] \\ \tilde{\mathcal{H}}_6 &= 8g_6 [8(1 - 3\Lambda)(1 - 4\Lambda)(1 - 5\Lambda)y^6 - 60(1 - 3\Lambda)(1 - 4\Lambda)y^4 + 90(1 - 3\Lambda)y^2 - 15]\end{aligned}$$

where the constants g_i , $i = 1, 2, \dots, 6$, take the values

$$g_1 = g_2 = 1, \quad g_3 = g_4 = (1 - \Lambda), \quad g_5 = g_6 = (1 - \Lambda)(1 - 2\Lambda).$$

Hence, the polynomials $\tilde{\mathcal{H}}_n$, obtained from the generating function $\mathcal{F}(t, y, \Lambda)$, are essentially the same ones that the polynomials \mathcal{H}_n , obtained from the Rodrigues formula. They coincide in the fundamental part (y -dependent polynomial part written between square brackets [...]) and only differ in the values of the global multiplicative coefficients that now are given by g_i instead of k_i (although g_i and k_i are rather similar nevertheless $g_i \neq k_i$).

The existence of this generating function allows us to establish the existence of Λ -dependent recursion relations among deformed Hermite polynomials of different orders.

Firstly, let us take the derivative of $\mathcal{F}(t, y, \Lambda)$ with respect to t and then multiply both sides by $1 + \Lambda(2ty - t^2)$. In this way we obtain the relation

$$2(y - t) \left(\sum_{k=0}^{\infty} \frac{1}{k!} \tilde{\mathcal{H}}_k(y, \Lambda) t^k \right) = (1 + \Lambda(2ty - t^2)) \left(\sum_{k=0}^{\infty} \frac{1}{k!} \tilde{\mathcal{H}}_{k+1}(y, \Lambda) t^k \right)$$

that leads to

$$\tilde{\mathcal{H}}_1 = 2y \tilde{\mathcal{H}}_0$$

and to the following recursion relation

$$\tilde{\mathcal{H}}_{n+1} = 2y(1 - n\Lambda) \tilde{\mathcal{H}}_n - n(2 - (n-1)\Lambda) \tilde{\mathcal{H}}_{n-1}, \quad n \geq 1.$$

It is clear that if we particularize this formula for $\Lambda = 0$ then we recover the recursion formula of the standard Hermite polynomials.

Finally, if we take the derivative of the function $\mathcal{F}(t, y, \Lambda)$ with respect to the variable y and then multiply both sides by $1 + \Lambda(2ty - t^2)$ we arrive to the following relation

$$2t(1 + \Lambda(2ty - t^2))^{1/\Lambda} = (1 + \Lambda(2ty - t^2)) \left(\sum_{k=0}^{\infty} \frac{1}{k!} \tilde{\mathcal{H}}'_k(y, \Lambda) t^k \right)$$

so that, equating the coefficients of each power of t , we obtain

$$\begin{aligned}\tilde{\mathcal{H}}'_0 &= 0 \\ \tilde{\mathcal{H}}'_2 + 4\Lambda y \tilde{\mathcal{H}}'_1 - 2\Lambda y \tilde{\mathcal{H}}'_0 &= 2\tilde{\mathcal{H}}_0\end{aligned}$$

as well as the general expression

$$\tilde{\mathcal{H}}'_{n+2} + (n+2)\Lambda [2y \tilde{\mathcal{H}}'_{n+1} - (n+1)\tilde{\mathcal{H}}'_n] = 2(n+2)\tilde{\mathcal{H}}_{n+1}, \quad n = 0, 1, 2, \dots$$

Once more we obtain for $\Lambda = 0$ a well known property of the standard Hermite polynomials.

6 Final comments and outlook

We have studied the quantum version of the λ -dependent one-dimensional non-linear oscillator of Mathews and Lakshmanan. The first difficulty to deal with was the question of the order ambiguity in the quantization of the Hamiltonian since it is a system with a position-dependent mass; this problem was solved by introducing a prescription (previously studied in Ref. [40]) obtained from the analysis of the properties of the classical system (existence of a Killing vector and a λ -dependent invariant measure). The quantum Hamiltonian obtained by this procedure has proved to belong to the restricted family of exactly solvable Hamiltonians. This is a very interesting situation: quantum systems with nonpolynomial rational potentials (and standard kinetical term) are hard to study and difficult to be solved and, on the other hand, quantum systems with a position-dependent mass are also difficult to study and generally not exactly solvable. The system we have considered combines however both traits in a very specific way, and this combination turns the problem exactly solvable.

We recall that has been proved in [3] that although the classical two-dimensional nonlinear oscillator is not separable in cartesian coordinates it admits however Hamilton-Jacobi separability in three other different coordinate systems (polar coordinates and two other λ -dependent systems). It is known that classical Hamilton-Jacobi separability and quantum Schrödinger separability are closely related, so it is to be expected that the $n = 2$ quantum system will also admit a solvable Schrödinger equation. We think that this question must be investigated, as well as its connections to other approaches related with the factorization of two-dimensional systems [55],[58],[60]. For instance, it was shown in [60] that the Calogero and Calogero–Sutherland models [61, 62] have the generalized shape invariance property.

It is interesting to remark the existence of another family of Hermite-related polynomials connected with the harmonic oscillator: the so called “relativistic Hermite polynomials” introduced in [63] in the study of quantum relativistic harmonic oscillators (see also [64] and [65] and references therein). Although our approach has been entirely non-relativistic and the “ λ -deformed Hermite” polynomials \mathcal{H}_n (or $\tilde{\mathcal{H}}_n$) seem to be rather different to these “relativistic Hermite polynomials”, the existence of some kind of relationship seems quite probable and would be worth studying.

Finally, also in Ref. [3], a geometrical interpretation of this nonlinear system was proposed (a similar interpretation is discussed in [66, 67] with λ in the kinetic term and a λ -independent potential). The main idea is that this λ -dependent system represents a nonlinear model for the harmonic oscillator in the circle S^1 ($\lambda < 0$) or in the hyperbolic line H^1 ($\lambda > 0$). Remark that in the $n = 1$ case, it is convenient to consider S^1 and H^1 as one-dimensional spaces obtained by endowing each single geodesic of S^2 or H^2 with the induced metric (motion on S^1 and H^1 will correspond to the $J = 0$ radial motions on S^2 or H^2). The point is that, if we assume that this geometric interpretation is correct, then the λ -dependent “deformed Hermite” polynomials \mathcal{H}_n (or $\tilde{\mathcal{H}}_n$) could be interpreted as the “curved” version of the Hermite polynomials on the circle S^1 or in the hyperbolic line H^1 . We think that this is another question to be studied.

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Figure Captions

- FIGURE I. Plot of $V_\lambda(x) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda < 0$, as a function of x , for $\lambda = -2$ (upper curve), and $\lambda = -1$ (lower curve).
- FIGURE II. Plot of $V_\lambda(x) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda > 0$, as a function of x , for $\lambda = 1$ (upper curve), and $\lambda = 2$ (lower curve).
- FIGURE III. Plot of the energy e_m as a function of m for $\Lambda = 0.30$ (lower curve) and $\Lambda = 0.15$ (upper curve). The curves also show the plot of the points (m, e_m) for the values $m = 0, 1, 2, 3$, and $m = 0, 1, \dots, 6$, respectively. For $\Lambda = 0.30$ there exist four bound states (four thick points in the curve) and for $\Lambda = 0.15$ seven bound states (seven thick points in the curve). When Λ decreases the maximum of the curve moves into the up right and in the limit $\Lambda \rightarrow 0$ the curve converges into a straight line parallel to the diagonal (dashed line).
- FIGURE IV. Plot of the energy e_m as a function of m for $\Lambda = 0.30$ (lower curve) and $\Lambda = -0.30$ (upper curve). The thick points (m, e_m) , corresponding to the values $m = 0, 1, 2, 3$, represent the four bound states existing for $\Lambda = 0.30$ and the first four bound states for $\Lambda = -0.30$. The straight line (dashed line) placed in the middle corresponds to the linear harmonic oscillator.

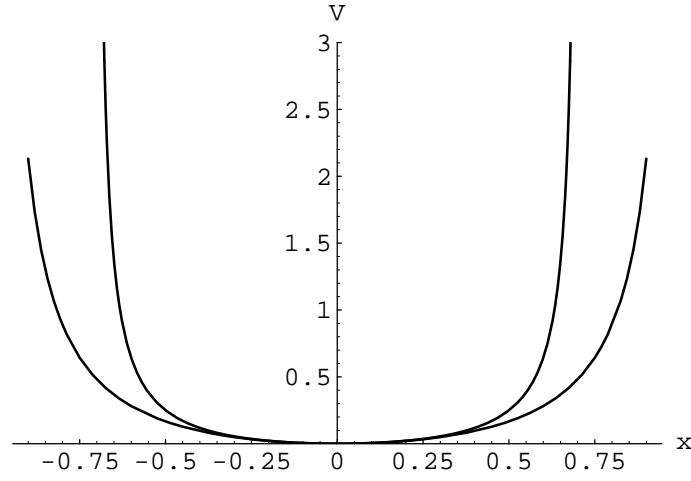


FIGURE I. Plot of $V_\lambda(x) = (1/2) (\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda < 0$, as a function of x , for $\lambda = -2$ (upper curve), and $\lambda = -1$ (lower curve).

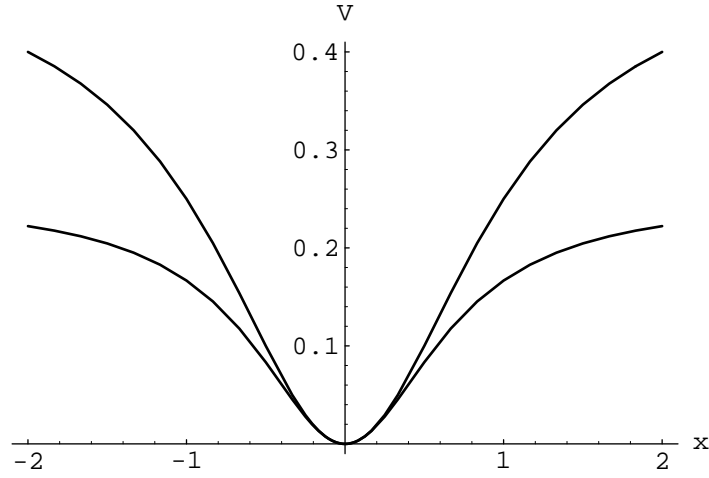


FIGURE II. Plot of $V_\lambda(x) = (1/2) (\alpha^2 x^2)/(1 + \lambda x^2)$, $\alpha = 1$, $\lambda > 0$, as a function of x , for $\lambda = 1$ (upper curve), and $\lambda = 2$ (lower curve).

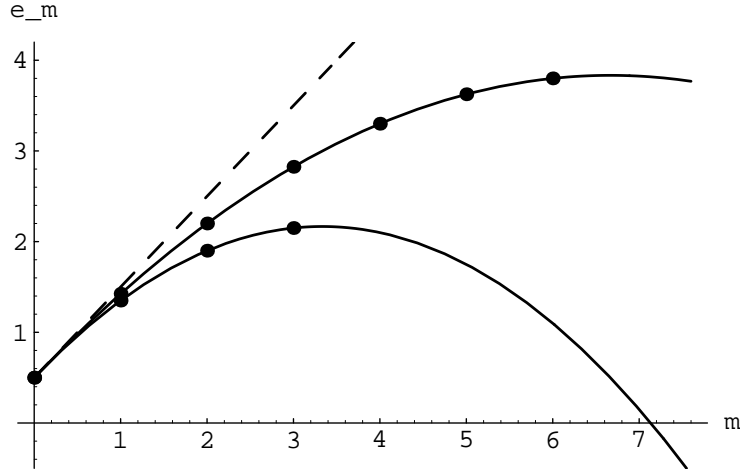


FIGURE III. Plot of the energy e_m as a function of m for $\Lambda = 0.30$ (lower curve) and $\Lambda = 0.15$ (upper curve). The curves also show the plot of the points (m, e_m) for the values $m = 0, 1, 2, 3$, and $m = 0, 1, \dots, 6$, respectively. For $\Lambda = 0.30$ there exist four bound states (four thick points in the curve) and for $\Lambda = 0.15$ seven bound states (seven thick points in the curve). When Λ decreases the maximum of the curve moves into the up right and in the limit $\Lambda \rightarrow 0$ the curve converges into a straight line parallel to the diagonal (dashed line).

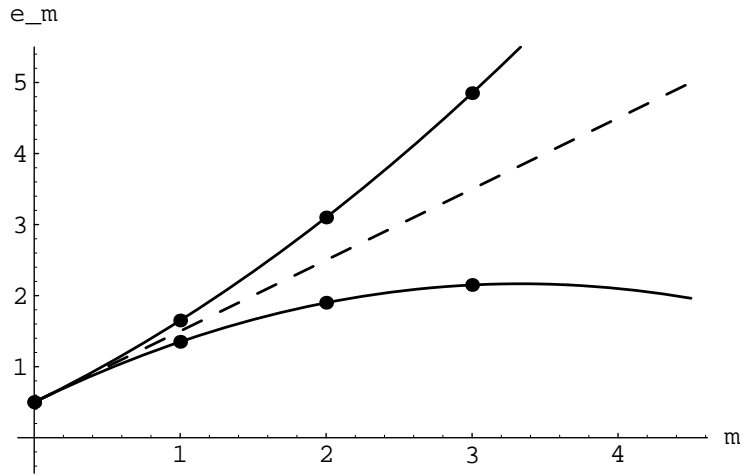


FIGURE IV. Plot of the energy e_m as a function of m for $\Lambda = 0.30$ (lower curve) and $\Lambda = -0.30$ (upper curve). The thick points (m, e_m) , corresponding to the values $m = 0, 1, 2, 3$, represent the four bound states existing for $\Lambda = 0.30$ and the first four bound states for $\Lambda = -0.30$. The straight line (dashed line) placed in the middle corresponds to the linear harmonic oscillator.